# TWO INTEGRAL REPRESENTATIONS

BY

G. A. EDGAR*

Department of Mathematics

The Ohio State University

Columbus, Ohio 43210

U.S.A.

## 0. INTRODUCTION

Proved here are two integral representation theorems related to complex variables. They illustrate one way in which Choquet's Theorem and its relatives can be used. Other examples, analyzed in a similar way, can be found in [10], [4], [2]. Both of the theorems proved here were originally proved by other methods, but here I have proved them by analyzing the extremal structure of an appropriate compact convex set.

We will use the following notation:

$$U = \{z \in \mathbf{C} : |z| < 1\}$$

$$U^+ = \{z \in U : \operatorname{Im} z > 0\}$$

$$U^- = \{z \in U : \operatorname{Im} z < 0\}.$$

The space of all holomorphic functions on $U$ will be denoted $H(U)$. It will be given the topology of uniform convergence on compact subsets of $U$. It is then a locally convex topological vector space. (In fact, a Fréchet space.) It has the property that every closed, bounded set in $H(U)$ is compact. (So $H(U)$ is a "Montel space".) This follows from Montel's theorem on normal families. See [8, p. 32]. We write ex $S$ for the set of extreme points of a convex set $S$.

## 1. THE RIESZ-HERGLOTZ REPRESENTATION

The following theorem was apparently first proved by F. Riesz, but is most commonly attributed to G. Herglotz.

THEOREM 1.1. If $f \in H(U)$ and $\operatorname{Re} f(z) > 0$ for all $z \in U$, then there is a positive, finite, Borel measure $\mu$ on $T = \{\alpha \in \mathbf{C}: |\alpha| = 1\}$ and a real number $y$ such that $f$ can be represented

$$(1) \qquad f(z) = iy + \int_T \frac{\alpha + z}{\alpha - z}\,\mu(d\alpha), \qquad z \in U.$$

* Supported in part by N.S.F. grant MCS 8003078.

By taking real parts, we can rewrite this as a representation for positive harmonic functions $h$ on $U$:

$$h(z) = \int_T \frac{1 - |z|^2}{|z - \alpha|^2}\, \mu(d\alpha).$$

The integrand is the Poisson kernel. A proof of Theorem 1.1 (not using Choquet's theorem) can be found, for example, in [9, p. 262].

Let $S = \{f \in H(U) : \text{Re } f(z) > 0 \text{ for } z \in U, f(0) = 1\}$. Then $S$ is closed, since by the minimum principle for the harmonic function Re $f$, $S$ is also equal to $\{f \in H(u) : \text{Re } f(z) \geq 0, f(0) = 1\}$. Clearly $S$ is convex. By Schwarz's Lemma, for $R < 1$ we have, for $f \in S$,

$$\max_{|z| \leq R} |f(z)| \leq \frac{1 + R}{1 - R}.$$

(See [1, p. 136, Exercise 2] for the appropriate version of Schwarz's Lemma.) This shows that $S$ is a bounded set. Therefore $S$ is a compact, convex set in the locally convex space $H(U)$. It remains to determine ex $S$. For $\alpha \in T$, let $h_\alpha(z) = \frac{\alpha + z}{\alpha - z}$.

Lemma 1.2. ex $S = \{h_\alpha : \alpha \in T\}$.

Lemma 1.2 can easily be deduced from Theorem 1.1 (see [10, p. 117]). But the point here is to prove Theorem 1.1 using Lemma 1.2. R. Phelps asked in [10] for a proof of Lemma 1.2 without using the integral representation. F. Holland [5] provided such a proof. I provide another one below.

Proof of 1.2. First, note that if $f \in S$, then $|f'(0)| \leq 2$, by Schwarz's lemma (see [1, p. 136]). Given $f \in S$, let $\alpha = \frac{1}{2}\overline{f'(0)} = a + ib$. Then $|\alpha| \leq 1$. Consider

$$g(z) = \left[-\frac{1}{2}a + \frac{1}{4}\left(z + \frac{1}{z}\right)\right] f(z) + \left[\frac{i}{2}b + \frac{1}{4}\left(z - \frac{1}{z}\right)\right].$$

Now $\lim_{z\to 0} g(z) = 0$, so $g$ is holomorphic on $U$. Let $f_1 = f + g$ and $f_2 = f - g$. I claim $f_1, f_2 \in S$. To prove this, we may assume $f$ is holomorphic on $\bar{U}$ by considering $f(rz)$, $r < 1$, and then letting $r \to 1$. Now for $z = e^{i\theta}$, we have Re $f(e^{i\theta}) \geq 0$, so

$$\text{Re } f_1(e^{i\theta}) = [1 - \tfrac{1}{2}a + \tfrac{1}{2}\cos\theta]\, \text{Re } f(e^{i\theta}) \geq 0.$$

By the maximum principle, Re $f_1(z) \geq 0$ on all of $U$. Therefore $f_1 \in S$. Similarly, $f_2 \in S$. Note also that $f = \frac{1}{2}(f_1 + f_2)$. So if $f$ is an extreme point of $S$, then $f_1 = f = f_2$, so $g \equiv 0$, so

$$f(z) = \frac{\frac{i}{2}\, b + \frac{1}{4}\,(z - \frac{1}{z})}{\frac{1}{2}\, a - \frac{1}{4}\,(z + \frac{1}{z})} = \frac{1 - 2ibz - z^2}{1 - 2az + z^2} .$$

Now repeat the process: consider

$$g(z) = \left[-\frac{1}{2}\, b + \frac{1}{4i}\,(z - \frac{1}{z})\right] f(z) + \left[\frac{1}{2i}\, a + \frac{1}{4i}\,(z + \frac{1}{z})\right] .$$

Then $\lim_{z\to 0} g(z) = 0$, so $g$ is holomorphic. Let $f_1 = f + g$, $f_2 = f - g$. As before, $f_1, f_2 \in S$. If $f$ is extreme, then $g \equiv 0$, so

$$f(z) = \frac{\frac{1}{2i}\, a + \frac{1}{4i}\,(z + \frac{1}{z})}{\frac{1}{2}\, b - \frac{1}{4i}\,(z - \frac{1}{z})} = \frac{1 + 2az + z^2}{1 + 2ibz - z^2}$$

Comparing the two expressions for $f(z)$ yields

$$\frac{1 - 2ibz - z^2}{1 - 2az + z^2} = \frac{1 + 2az + z^2}{1 + 2ibz - z^2}$$

or

$$1 + z^2(-2 + 4b^2) + z^4 = 1 + z^2(2 - 4a^2) + z^4$$

so $a^2 + b^2 = 1$, and $|\alpha| = 1$. Then

$$f(z) = \frac{1 + 2az + z^2}{1 + 2ibz - z^2} = \frac{1 + (\alpha + \bar{\alpha})\, z + z^2}{1 + (\alpha - \bar{\alpha})\, z - z^2}$$

$$= \frac{(\alpha + z)\, (\bar{\alpha} + z)}{(\alpha - z)\, (\bar{\alpha} + z)} = \frac{\alpha + z}{\alpha - z} .$$

This shows $\mathrm{ex}\, S \subseteq \{h_\alpha : |\alpha| = 1\}$.

Now there is at least one extreme point. Rotations of the disk map one function $h_\alpha$ into all the others, so if one is extreme, they all are. So $\mathrm{ex}\, S = \{h_\alpha : |\alpha| = 1\}$. ∎

Proof of 1.1. The map $\alpha \to h_\alpha$ is continuous from $T$ into $H(U)$. It is also injective. Since $T$ is compact, this map is a homeomorphism of $T$ onto $\mathrm{ex}\, S$. So $\mathrm{ex}\, S$ is a closed subset of $S$. By Choquet's Theorem (or even the Krein-Milman Theorem, see [10, Prop. 1.2]), for any $f \in S$ there is a probability measure $\mu_1$ on $\mathrm{ex}\, S$ such that

$$f = \int_S h\, \mu_1(dh) .$$

By means of the homeomorphism described above, the measure $\mu_1$ corresponds to a measure $\mu$ on T such that

$$f = \int_T h_\alpha \, \mu(d\alpha).$$

Since, for $z \in U$, the map $f \to f(z)$ is a continuous linear functional, we may conclude that

$$f(z) = \int_T h_\alpha(z) \, \mu(d\alpha)$$

for all $z \in U$. If $f$ is any holomorphic function on U with nonnegative real part, then $f = cf_1 + iy$ for $c = \mathrm{Re}\, f(0)$, $f_1 \in S$, $y \in \mathbb{R}$, so such an $f$ can be represented in the form (1), where $\mu$ is $c$ times a probability measure. ∎

## 2. ROBERTSON'S REPRESENTATION

The following is a definition of Rogosinski [6].

Definition. A holomorphic function $f$ defined on U is called *typically real* if $f(z)$ is real if and only if $z$ is real.

Robertson [7] proved the following representation theorem for typically real functions.

THEOREM 2.1. Let $f \in H(U)$ be a typically real function. Then there exist real numbers a, b and a probability measure $\mu$ on $[-1, 1]$ such that

$$f(z) = a + b \int_{-1}^{1} \frac{z}{1 + 2tz + z^2} \mu(dt), \quad z \in U.$$

Robertson proved this by transforming the problem into another one where an integral representation theorem was already known. In this paper, we will analyse the extreme point structure of an appropriate convex set. Fewer details will be given than in the previous case.

Let

$$T = \{f \in H(u): f \text{ is typically real, } f(0) = 0,\ f'(0) = 1\}.$$

Note that if $f_1$ is typically real, then $f_1 = a + bf$ where a, b are real and $f \in T$. As before, $T$ is a closed, convex set in $H(U)$. Using the Schwarz theorem, it can be shown that if $f \in T$ and $R < 1$, then

$$\max_{|z| \le R} |f(z)| \le \frac{R}{(1 - R)^2}.$$

Therefore $T$ is bounded, and hence compact.

For $t \in [-1, 1]$, let

$$h_t(z) = \frac{z}{1 + 2tz + z^2}$$

for all $z \in U$. As in the proof of Theorem 1.1, the only remaining step in the proof of Theorem 2.1 is the following lemma.

Lemma 2.2. $\operatorname{ex} T = \{h_t : t \in [-1, 1]\}$.

Of course, this is an easy consequence of Theorem 2.1. (See [3] for an explicit statement of this fact.) But our intention is to use 2.2 to prove 2.1. Another possibility is to transform the convex set $T$ affinely into a convex set whose extreme points are known. (This is Robertson's approach.) But our intention is to proceed directly with $T$.

Proof of 2.2. We first claim that if $f \in T$ and $A > 2$, then

$$f_1(z) = (A - \frac{1}{z} - z)\, f(z)$$

and

$$f_2(z) = (A + \frac{1}{z} + z)\, f(z)$$

are typically real. By considering $f(rz)/r$, $0 < r < 1$, as $r \to 1$, we may assume $f$ is holomorphic on $\bar{U}$. For $-1 \le x \le 1$, $x \ne 0$, $f_1(x)$ and $f_2(x)$ are real. And $f_1(0) = -1$, $f_2(0) = 1$ are also real. For $z = e^{i\theta}$, $\operatorname{Im} f_1(e^{i\theta}) = (A - 2\cos\theta) \operatorname{Im} f(e^{i\theta})$ has the same sign as $\operatorname{Im} f(e^{i\theta})$. So $\operatorname{Im} f_1$ is positive on $U^+$, negative on $U^-$. Therefore $f_1$, $f_2$ are typically real.

Now suppose $f \in \operatorname{ex} T$. Let $t = -f''(0)/4$. I will show $f = h_t$ and $-1 \le t \le 1$. Choose $A > 2|t| + 2$. Let

$$g(z) = \frac{1}{A}(1 - (\frac{1}{z} + z + 2t)\, f(z)).$$

Note that $g(0) = g'(0) = 0$. Now

$$(f + g)(z) = \frac{1}{A} + \frac{1}{A}(A - 2t - \frac{1}{z} - z)\, f(z)$$

is typically real, since $A - 2t > 2$. Similarly, $f - g$ is typically real since $A + 2t > 2$. Also $(f + g)(0) = (f - g)(0) = 0$, $(f + g)'(0) = (f - g)'(0) = 1$. So $f + g$, $f - g \in T$. But $f$ is extreme, so $g \equiv 0$, and thus

$$f(z) = \frac{z}{1 + 2tz + z^2}.$$

If $t > 1$ or $t < -1$, this has a pole inside $U$. So $-1 \leq t \leq 1$. This shows that $\mathrm{ex}\, T \subseteq \{h_t: -1 \leq t \leq 1\}$.

For fixed $a \in [-1, 1]$, the map $T_a$, defined by $T_a(f) = \mathrm{Re}\,(-12af''(0) - f'''(0))$, is a continuous linear functional on $H(U)$, so its maximum on $T$ is achieved at an extreme point. But $T_a(h_t) = -24t^2 + 48at + 6$ has its maximum at $t = a$, so $h_a \in \mathrm{ex}\, T$, $a \in [-1, 1]$. Therefore $\mathrm{ex}\, T = \{h_t: -1 \leq t \leq 1\}$. ∎

The proof of Theorem 2.1 from Lemma 2.2 is the same as the corresponding proof of Theorem 1.1, and is therefore omitted.

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